SOME PROBLEMS CONCERNING NONSTATIONARY CURRENTS IN DENSE PLASMA SYSTEMS CONFINED BY WALLS

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Nonstationary currents are examined in a dense magnetized plasma with $\beta \gg 1$, in which energy release and heat loss by thermal conduction and radiation are possible. Solutions are found in two limiting cases: $|f| \gg |\operatorname{div}(\eta \nabla T)|$ and $|f| \ll |\operatorname{div}(\eta \nabla T)|$ (f is the radiation intensity, η is the coefficient of heat conduction, and T is the temperature). In the first case a solution was obtained of some problems of the cooling and heating of a plasma illustrated in part by the evolution in time of the temperature profile in the boundary layer. In the second case an isomorphic solution was found for an arbitrary dependence of the coefficient of heat conduction on the temperature, pressure, and magnetic field.

1. Fundamental Equations. One-dimensional nonstationary currents of a fluid (plasma) are examined below, which are described in general by the following system of equations [1, 2]:

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)=-\frac{\partial}{\partial x}\left(p+\frac{H^2}{8\pi}\right) \tag{1.1}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0 \tag{1.2}$$

$$\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}(Hv)=0 \tag{1.3}$$

$$\frac{\gamma p}{(\gamma-1)T}\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x}\right)-\left(\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial x}\right)=\frac{\partial}{\partial x}\left(\eta\frac{\partial T}{\partial x}\right)-f+g \tag{1.4}$$

$$p=R\rho T \tag{1.5}$$

Here p(x, t), ρ(x, t), T(x, t), v(x, t), H(x, t), x, and t are the pressure, density, temperature, velocity, magnetic field, coordinate, and time, respectively; γ is an adiabatic index; R is the gas constant; $\eta(T, p, H)$ is the coefficient of heat conduction, f(p, T) and g(p, T, x) are the intensities of radiation and energy release. It is assumed that the magnetic field, perpendicular to x, is frozen into the plasma. At the point x = 0 the plasma is in contact with the boundary wall (plane), where the following conditions are satisfied:

$$x=0,\quad v=0,\quad T=T_0 \tag{1.6}$$

The system of equations (1.1)-(1.5) describes the processes of heating and cooling under the conditions of a thermonuclear reactor with a dense plasma which is confined by walls [1]. A characteristic property of these processes is their slowness in comparison with the time of circulation of sound waves. For example, the time of passage of a sound wave across a plasma column 100 cm in diameter with T = 10^{8}°K will be on the order of 10^{-6} sec, whereas the characteristic time of plasma heating is on the order of 10^{-4}-10^{-3} sec, and the cooling time is on the order of several tens of microseconds. In view of this one can calculate that the pressure along the x axis is able to equalize

$$\frac{\partial}{\partial x}\left(p+\frac{H^2}{8\pi}\right)=0 \tag{1.7}$$

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Furthermore, in systems where the plasma is confined by magnetic walls the pressure is usually small in comparison with the gas kinetics

$$p \gg H^2 / (8\pi) \tag{1.8}$$

and the magnetic field has an effect only on the heat conduction.

Since in general analytical solutions of the system (1.1)-(1.5) cannot be achieved, we will examine below two limiting cases: where heat conduction is low and the processes of radiation and energy release dominate

$$\left|\frac{\partial}{\partial x}\left(\eta \frac{\partial T}{\partial x}\right)\right| \ll |f-g| \tag{1.9}$$

and vice versa, where the role of heat conduction predominates

$$\left|\frac{\partial}{\partial x}\left(\eta \frac{\partial T}{\partial x}\right)\right| \gg |f-g| \tag{1.10}$$

2. Case of Dominant Role of Radiation and Energy Release. Dropping the heat conduction term in (1.4) in accordance with the condition (1.9) and the magnetic pressure term in (1.7) [substituting (1.1)] in accordance with (1.8), we see that the magnetic field now has no effect in general on the dynamics of the process. Taking (1.5) and (1.7) into account, Eqs. (1.2) and (1.4) are written in the form

$$\frac{\partial}{\partial t}\left(\frac{p}{T}\right) + p\frac{\partial}{\partial x}\left(\frac{v}{T}\right) = 0 \tag{2.1}$$

$$\frac{\gamma}{(\gamma-1)}\frac{p}{T}\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x}\right) - \frac{\partial p}{\partial t} = -f + g \tag{2.2}$$

In the present case it is expedient to write them in Lagrangian coordinates. The independent variables will be t and $a \equiv \xi(t=0)$, while the values sought are $T(a, t)$, $p(a, t)$, and $\xi(a, t)$, where ξ are coordinates of the Langrangian points. Equations (1.7), (2.1), and (2.2) are rewritten in the form

$$\frac{\partial p}{\partial a} = 0 \tag{2.3}$$

$$\frac{\partial \xi}{\partial a} - \frac{T(a,t)}{T_0(a)}\frac{p_0}{p(t)} = 0; \quad T_0(a) = T(a, t=0), \quad p_0 = p(t=0) \tag{2.4}$$

$$\frac{\gamma}{\gamma-1}\frac{p}{T}\frac{\partial T}{\partial t} - \frac{dp}{\partial t} + f - g = 0 \tag{2.5}$$

Equations (2.3)-(2.5) are easily solved since f and g are functions of p and T and do not depend on ξ, a, and t. This condition holds for problems involving nonstationary flow through a thermonuclear reactor (after heating) and cooling of a plasma by radiation. It is especially simple to find the solution when the pressure does not change with time: $p = p_0 = \text{const}$. This condition is of interest for problems of plasma cooling, since in view of the strong dependence of the reaction rate on temperature a considerable decrease in the energy release can occur with even a small decrease in T_{max}, and consequently in p.

If p = const, Eq. (2.5) is immediately integrated;

$$t = F(T(a,t)) - F(T_0(a,t)), \quad F(t) = -\frac{\gamma p_0}{\gamma-1}\int \frac{dT}{T(f-g)} \tag{2.6}$$

Substituting the $T(a, t)$ found from this into (2.4) and integrating it, we find $\xi(a, t)$.

We examine the case where g = 0 and $f = bT^{-1}$, and the rest of the problem is solved analytically. We then find (2.7) from (2.6),

$$T = -\frac{(\gamma-1)}{\gamma}\frac{b}{p_0}t + T_0(a) \tag{2.7}$$

We take the initial profile in the form $T_0(a) = k\sqrt{a}$. We then obtain from (2.4) and (2.7)

$$\xi = a - 2\lambda t\sqrt{a}, \qquad \lambda = \frac{(\gamma-1)b}{\gamma k p_0} \tag{2.8}$$

Expressing a through ξ and t and substituting into (2.7), we obtain, returning to Eulerian coordinates ($\xi \equiv x$),

$$T(x, t) = k(x + 2\lambda^2 t^2 + 2\lambda t \sqrt{\lambda^2 t^2 + x})^{1/2} - \lambda k t \tag{2.9}$$

By virtue of (2.10) this expression is valid for

$$0 \leqslant x \leqslant l - 2\lambda t \sqrt{l}, \quad 0 \leqslant t \leqslant \frac{\sqrt{l}}{2\lambda}$$

where l is the thickness of the plasma layer at the initial moment.

It is seen from (2.9) in particular how the steepness of the profile T(x, t) changes with time. One can show that $\partial T/\partial x$ in the given case grows not more than twofold, despite the opinion sometimes expressed that deexcitation of a plasma confined by walls leads to a considerable increase in the steepness of the front.

In some cases one can obtain a solution of the system (2.3)-(2.5) without making the assumption that p = const. We examine the case $g = 0$, $f = bp^2T^{-3/2}$ (bremsstrahlung). We write Eq. (2.5) in the form

$$\frac{\partial z}{\partial y} = -b\left(\frac{dy}{dt}\right)^{-1}(ry - sz)$$

$$y = \ln p, \quad z = \frac{\gamma}{\gamma - 1}\ln T - \ln p, \quad r = \frac{3 - \gamma}{2\gamma}, \quad s = \frac{3(\gamma - 1)}{2\gamma}$$

Integrating this expression and returning to the previous notation, we obtain

$$T^{3/2} = sp^s\left\{-b\int_0^t p^{-r}dt + \varphi(a)\right\} \tag{2.10}$$

$$\varphi(a) = T_0^{3/2}(a)(sp^s)^{-1}$$

On the other hand, the plasma pressure is proportional to its internal energy, so that

$$\frac{dp}{dt} = -\frac{(\gamma - 1)}{l}\int_0^l f dx = -\frac{(\gamma - 1)bp_0 p}{l}\int_0^l \frac{T^{-1/2}(a, t)}{T_0(a)}da \tag{2.11}$$

We define

$$\zeta = \int_0^t p^{-r}dt, \qquad p = \left(\frac{d\zeta}{dt}\right)^{-1/r} \tag{2.12}$$

Substituting these expressions into (2.10) and then $T(\zeta)$ and $p(\zeta)$ into (2.11), we find a differential equation for determining $\zeta(t)$,

$$\frac{d^2\zeta}{dt^2} = -A\left(\frac{d\zeta}{dt}\right)^{1-s/3r}F(\zeta) \tag{2.13}$$

$$A = (\gamma - 1)rs^{-1/3}bp_0/l, \quad F(\zeta) = \int_0^l \frac{(\varphi(a) - b\zeta)^{-1/3}}{T_0(a)}$$

Integrating (2.13) we obtain

$$t = \int \frac{d\zeta}{\left[-2A(3-\gamma)^{-1}\left(\int F(\zeta)\,d\zeta + C_1\right)\right]^{1/2(3-\gamma)}} + C_2 \tag{2.14}$$

where C_1 and C_2 are constants of integration. Equations (2.10), (2.12), and (2.14) give a solution to the problem proposed.

Under these conditions, where $g \neq 0$ and is a function of x (for example, in heating a plasma by outside sources of heat), the solution of (2.3)-(2.5) is somewhat complicated. However, the fact that in cases of practical interest the dependence g(x) is quite simple is a simplifying circumstance. For example,

$$g = \begin{cases} 0, & 0 \leqslant x < x_2 = l_0 - l_1 \\ g_0, & x_2 \leqslant x \leqslant x_3 \equiv l_0 \end{cases} \tag{2.15}$$

Here l_0 is the half width of a plasma layer with plane symmetry at the point x_1; l_1 is the half width of the zone of heating.

In different regions of variation in x one can find solutions as was done above and then join them together.

We present the solution of a very simple problem of a similar type – the heating and expansion of a gas where g(x) has the form of (2.15) and radiation is absent ($f=0$). In this case there will be solutions of three different types, corresponding to the plasma particles which are always in the zone of heating, to the particles first found within this zone and then escaping from it, and for the particles outside the heating zone. Turning to the discovery of a solution of the first type, keeping in mind (2.15), we express g through dp/dt from (2.11) (where g must now be used in place of f) and substitute in (2.5). We obtain

$$\frac{\gamma}{(\gamma-1)}\frac{\partial T}{\partial t}=\frac{(\gamma-1)\,l_1+l_0}{(\gamma-1)\,l_0}\frac{1}{p}\frac{dp}{dt}$$

Hence

$$T=T_0\left(\frac{p}{p_0}\right)^{\alpha},\qquad \alpha=\frac{(\gamma-1)\,l_1+l_0}{(\gamma-1)\,l_0} \tag{2.16}$$

This solution holds for the region $l_0-l_1\leq x\leq l_0$. In the next case the plasma particles while they are found within the heating zone are heated in accordance with Eq. (2.16) up to some temperature $T_1=T_0(p_1/p_0)^{\alpha}$, and then from T_1 to T according to the adiabatic equation $T=T_1(p/p_1)^{(\gamma-1)/\gamma}$. Consequently,

$$\frac{T}{T_0}=\left(\frac{p_1}{p}\right)^{\alpha-(1-\gamma)/\gamma}\left(\frac{p}{p_0}\right)^{\alpha} \tag{2.17}$$

If particles having temperature T are found at point x, then

$$\frac{l_0-x}{l_0-l_1}=\left(\frac{p_1}{p}\right)^{1/\gamma}$$

From this, expressing p_1/p through x and substituting in (2.17), we obtain a solution of the second type,

$$\frac{T}{T_0}=\left(\frac{p}{p_0}\right)^{\alpha}\left(\frac{l_0-x}{l_0-l_1}\right)^{\alpha\gamma+1-\gamma}$$

The region of its application is

$$x_3\leqslant x\leqslant l_0-l_1,\qquad x_3=l_0-(l_0-l_1)\,(p/p_0)^{-1/\gamma}$$

Finally, the solution of the third type has the form

$$\frac{T}{T_0}=\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}$$

and holds for $0\leq x<x_3$.

3. Case of Dominant Role of Heat Conduction. If condition (1.10) is satisfied, then in a number of problems isomorphic solutions of system (1.1)-(1.5) can be found. We assume that condition (1.7) is satisfied and at the same time

$$\frac{\partial}{\partial t}\left(p+\frac{H^2}{8\pi}\right)=0,\qquad p+\frac{H^2}{8\pi}=p_0=\text{const} \tag{3.1}$$

It is assumed that condition (1.8) is not satisfied, which is physically possible under the conditions of the problem under examination (confinement by walls), since the plasma with the field frozen-in shifts to the wall and is cooled there. In the given case Eq. (1.4) will have the form

$$\frac{\gamma}{\gamma-1}\frac{p}{T}\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x}\right)-\left(\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial x}\right)=\frac{\partial}{\partial x}\left(\eta\frac{\partial T}{\partial x}\right) \tag{3.2}$$

while we take Eq. (1.2) in the form (2.1). The process as a whole is described by Eqs. (3.1), (3.2), (2.1), and (1.3) with the boundary conditions (1.6).

Assuming that the temperature is expressed in energetic units (ergs), we introduce the dimensional values T_0, p_0, H_0, $S=T_0\eta(T_0, p_0, H_0)/p_0$ and form the dimensionless parameters

$$\tau = \frac{x}{\sqrt{St}}, \quad \theta = \frac{T}{T_0}, \quad \sigma = \frac{p}{p_0}, \quad h = \frac{H}{\sqrt{8\pi p_0}}, \quad u = \frac{tv}{x}, \quad \varkappa = \frac{\eta(T, p, H)}{\eta_0(T_0, p_0, H_0)}$$

Equations (3.1), (3.2), (2.1), and (1.3) are then written in the form

$$\sigma + h^2 = 1 \tag{3.3}$$

$$\tau\left(u - \frac{1}{2}\right)\left(\frac{\gamma}{\gamma - 1}\frac{\sigma}{\theta}\frac{d\theta}{d\tau} - \frac{d\sigma}{d\tau}\right) = \frac{d}{d\tau}\left(\varkappa\frac{d\theta}{\partial\tau}\right) \tag{3.4}$$

$$\tau\left(u - \frac{1}{2}\right)\frac{d}{d\tau}\left(\frac{\sigma}{\theta}\right) + \frac{\sigma}{\theta}\frac{d}{d\tau}(\tau u) = 0 \tag{3.5}$$

$$\tau\left(u - \frac{1}{2}\right)\frac{dh}{d\tau} + h\frac{d}{d\tau}(\tau u) = 0 \tag{3.6}$$

Thus, the problem comes down to the solution of a system of ordinary differential equations and the corresponding isomorphic solution. We emphasize that the isomorphism is preserved for any dependence of η on T, p, and H, which is made possible thanks to the form of the parameters θ, δ, and h indicated above. (The quantities x and t did not enter into them.*)

From (3.5), (3.6), and (3.3) we find

$$h = C\frac{\sigma}{\theta}, \quad \theta = \frac{C\sigma}{\sqrt{1-\sigma}}$$

where C is an arbitrary constant.

Equations (3.4) and (3.5) are now written thus:

$$\tau\left(u - \frac{1}{2}\right)\frac{d\sigma}{d\tau} - \frac{C(\gamma - 1)(1-\sigma)}{\gamma\sigma + 2(1-\sigma)}\frac{d}{d\tau}\left[\varkappa(2-\sigma)(1-\sigma)^{-3/2}\frac{d\sigma}{d\tau}\right] = 0 \tag{3.7}$$

$$\tau\left(u - \frac{1}{2}\right)\frac{d\sigma}{d\tau} - 2(1-\sigma)\frac{d}{d\tau}(u\tau) = 0 \tag{3.8}$$

We find a solution near $\tau = 0$, trying to satisfy the boundary condition (1.6), which now has the form

$$\sigma = 0, \quad \tau u = 0 \quad \text{for} \quad \tau = 0 \tag{3.9}$$

We assume that

$$\varkappa(\theta, \sigma, h) = \theta^l \sigma^m h^n \tag{3.10}$$

where l, m, and n are constants. A dependence of type (3.10) is characteristic for a magnetized plasma if its temperature is much higher than the magnetization temperature. We assume that near $\tau = 0$

$$\sigma(\tau) = \sigma_0\tau^\alpha, \quad u = u_0\tau^\beta \tag{3.11}$$

To satisfy condition (3.9) it is necessary that

$$\alpha > 0, \quad \beta + 1 > 0 \tag{3.12}$$

From (3.8) we get

$$-\alpha\sigma_0\tau^\alpha(u_0\tau^\beta - 1/2) + u_0(1+\beta)\tau^\beta = 0$$

This equation is satisfied at $\tau \to 0$ if we set

$$\beta = \alpha, \quad u_0 = -\alpha\sigma_0 / 2(\alpha + 1)$$

Neglecting the small terms in Eq. (3.7) we write it in the form

$$\tau\frac{d\sigma}{d\tau} + 2C(\gamma - 1)\frac{d}{d\tau}\left(\varkappa\frac{d\sigma}{d\tau}\right) = 0$$

Substituting here (3.10) and (3.11), we obtain

$$\alpha\sigma_0\tau^\alpha + 2C(\gamma - 1)\alpha[(m + l + 1)\alpha - 1]\tau^{(m+l+1)\alpha - 2} = 0 \tag{3.13}$$

* The isomorphic solution of the heat conduction equation for the arbitrary dependence of $\varkappa(T)$ was demonstrated by N. A. Dmitriev [3].

If we set

$$\alpha = (m + l + 1)^{-1} \tag{3.14}$$

and if

$$m + l + 1 > 0 \tag{3.15}$$

then the first term of (3.13), proportional to the positive power of τ as $\tau \to 0$, will be small in comparison with the second term, proportional to τ^{-1}. Since the coefficient of the second term for the condition (3.14) reverses at zero, this means that Eq. (3.13) is satisfied by the expression $\sigma = \sigma_0 \tau^{\alpha}$ in the limit $\tau \to 0$. In observance of (3.15) condition (3.12) is also satisfied, so that the solution found satisfies the boundary conditions.

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